# Analysis of Newton's Forward Interpolation Formula

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***Abstract:*** This work presents a theoretical analysis Newton's Forward Interpolation Formula. In order to analyze the method, unit step, unit ramp and sinusoidal signals are chosen. Also to check the performance of the considered method an increasing function and a decreasing function has considered. Some sampled values of a signal are calculated and then the Newton's forward Interpolation formula is used to reconstruct the signal such as image resizing. Errors are analyzed by comparing the actual sampled values with the values obtained by Newton's Interpolation formula.

**Keywords:** Interpolation, forward or interpolation formula, forward difference table, increasing and decreasing function, unit step, unit ramp.

## 1. Introduction

From very ancient time Interpolation is being used for various purposes. Sir Edmund Whittaker, a professor of Numerical Mathematics at the University of Edinburgh from 1913 to 1923, observed "the most common form of interpolation occurs when we seek data from a table which does not have the exact values we want." Liu Zhuo used the equivalent of second order Gregory-Newton interpolation to construct an "Imperial Standard Calendar". In 625 AD, Indian astronomer and mathematician Brahmagupta introduced a method for second order interpolation of the sine function and, later on, a method for interpolation of unequal-interval data. Numerous researchers study the possibility of Interpolation based on the fourier transformer, the Hartley transformer and the discrete cosine transform. In 1983, Parker et al. published a first comparison of Interpolation techniques in medical image processing. They failed, however to implement cubic B-spline Interpolation correctly and arrive at erroneous conclusions concerning this technique [1]. In present days, several algorithms are used for image resizing [2] based on Newton's Interpolation Formula [3].

In this paper, Newton's forward Interpolation formula for equal interval is used for reconstructing a signal defined by a function to analyze its performance. This paper is organized as follows: In section 2, I explain the mathematical principal of Newton's Interpolation method. The implementation is devised in section 3. In section 4 the experimental results are given. The conclusion is summarized in section 5.

## 2. Theory of Interpolation and the considered functions

### 2.1 Forward Interpolation Formula:

Newton's forward difference formula is a finite difference identity giving an interpolated value between tabulated points in terms of the first value and the powers of the forward difference Δ. Let $y = f(x)$ denotes a function which takes the values $y_0, y_1, y_3, \ldots\ldots, y_n$ for the equidistant values $x_0, x_1, x_2, \ldots\ldots, x_n$ respectively of the independent variable $x$. According to Newton's forward interpolation formula the value of $y$ at a particular point can be found as follows:

$$f(x) = g(u) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + . \frac{u(u-1)(u-2)\ldots\ldots(u-n+1)}{n!}\Delta^n y_0 \quad \ldots\ldots\ldots\ldots (1)$$

where $u = \dfrac{x - x_0}{h}$ and $h = x_1 - x_0$ . Also $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots\ldots\ldots$ are the 1st, 2nd, 3rd forward difference respectively and so on.

The forward difference is a finite difference defined by

$$\Delta a_n = a_{n+1} - a_n$$

Higher order differences are obtained by repeated operations of the forward difference operator,

$$\Delta^k a_n = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_{n,}$$

so

$$\begin{aligned}\Delta^2 a_n &= \Delta_n^2 \\ &= \Delta(\Delta_n)\end{aligned}$$

$$= \Delta(a_{n+1} - a_n)$$
$$= \Delta_{n+1} - \Delta_n$$
$$= a_{n+2} - 2a_{n+1} + a_n$$

In general,

$$\Delta_n^k \equiv \Delta^k a_n \equiv \sum_{i=0}^{k}(-1)^i \binom{k}{i} a_{n+k-i} \ldots \ldots \ldots (2)$$

**2.2 Unit step function:**

The unit step function is defined as:

$$U(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases} \ldots \ldots \ldots \ldots \ldots (3)$$

**2.3 Unit ramp function:**

The *ramp function*, denoted by *r* (*t*) is a signal whose amplitude increases proportionally as time increases. The mathematical definition of a ramp signal is

$$r(t) = \begin{cases} kt & t \geq 0 \\ 0 & t < 0 \end{cases} \ldots \ldots \ldots \ldots \ldots \ldots (4)$$

## 3. Analysis and implementation

To analyze the effect of the considered method and unit step [4] function is considered first. According to the definition the calculated value of this function using equation (3) is shown in the tabular form:

| Time (T) | Amplitude (x) |
|---|---|
| 0 | 1 |
| 2 | 1 |
| 4 | 1 |
| 6 | 1 |
| 8 | 1 |
| 10 | 1 |

Table 1: Value of an unit step function according to the independent variable T.

The unknown values are also calculated by using equation (1) to reconstruct the signal more accurately and are shown below:

| T | x | $\Delta x$ | $\Delta^2 x$ | $\Delta^3 x$ | $\Delta^4 x$ | $\Delta^5 x$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 0 | |
| 4 | 1 | 0 | 0 | 0 | | |
| 6 | 1 | 0 | 0 | | | |
| 8 | 1 | 0 | | | | |
| 10 | 1 | | | | | |

Table 2: Forward Difference table for calculating the unknown values of the function

| Time (T) | Amplitude (x) |
|---|---|
| 1 | 1 |
| 3 | 1 |
| 5 | 1 |
| 7 | 1 |
| 9 | 1 |

Table 3: Calculated values of unit step function using Newton's Forward Interpolation Formula

With these two sets of data a graph is plotted using MATLAB to compare those sets of values:

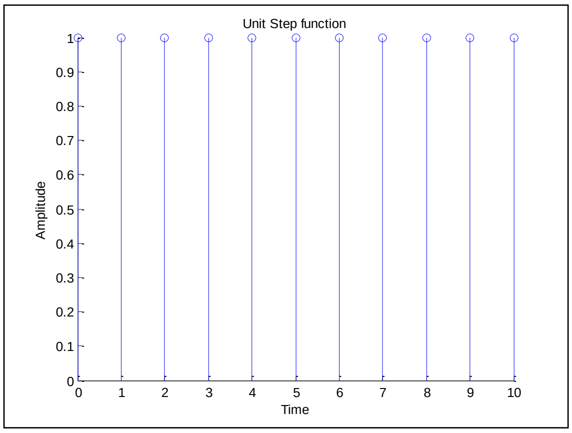


Figure 1: Unit step function

From this graph it is observed that in this case the considered method gives 100% accuracy as expected.
For a unit ramp function the calculated using equation (4) values using equation 4 are tabulated below:

| Time (T) | Amplitude (x) |
|---|---|
| 1 | 1 |
| 3 | 3 |
| 5 | 5 |
| 7 | 7 |
| 9 | 9 |

Table 4: Values for unit ramp function obtained from equation (4)

The values obtained by necessary calculations are given below in tabular form in Table 4 and Table 5 and a graph is also plotted using MATLAB shown in Figure 2:

| T | x | $\Delta x$ | $\Delta^2 x$ | $\Delta^3 x$ | $\Delta^4 x$ | $\Delta^5 x$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 | |
| 4 | 4 | 2 | 0 | 0 | | |
| 6 | 6 | 2 | 0 | | | |
| 8 | 8 | 2 | | | | |
| 10 | 10 | | | | | |

Table 4: Forward Difference table for unit ramp function

| Time (T) | Amplitude (x) |
|---|---|
| 1 | 1 |
| 3 | 3 |
| 5 | 5 |
| 7 | 7 |
| 9 | 9 |

Table 5: Calculated values of unit ramp function using Newton's Forward Interpolation Formula

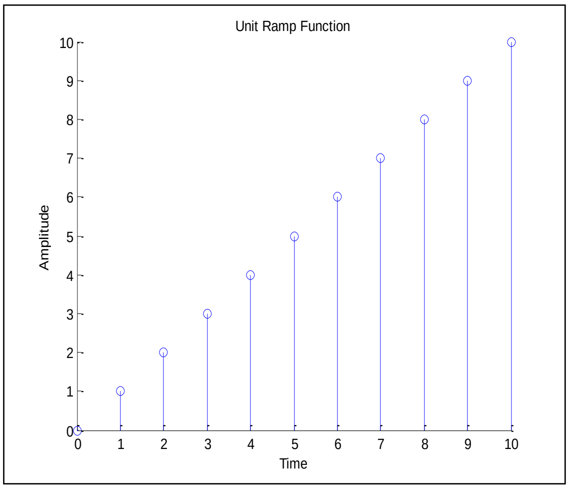


Figure 2: Graph of a unit ramp function

Doing the same procedure in the case of an increasing function (in this case, unit ramp function) it is clear that this method gives 100% accuracy.

But what will happen if we consider a decreasing function? The forward difference table is given below (Table 6):

| T | x | $\Delta x$ | $\Delta^2 x$ | $\Delta^3 x$ | $\Delta^4 x$ | $\Delta^5 x$ |
|---|---|---|---|---|---|---|
| 0 | 10 | -2 | 0 | 0 | 0 | 0 |
| 2 | 8 | -2 | 0 | 0 | 0 | |
| 4 | 6 | -2 | 0 | 0 | | |
| 6 | 4 | -2 | 0 | | | |
| 8 | 2 | -2 | | | | |
| 10 | 0 | | | | | |

Table 6: Forward Difference Table for a linearly decreasing function

Unknown values are obtained from Newton's polynomial. Those are tabulated below in Table 7:

| Time (T) | Amplitude (x) |
|---|---|
| 1 | 19 |
| 3 | 17 |
| 5 | 15 |
| 7 | 13 |
| 9 | 11 |

Table 7: calculated values of the function

To see the effect we can draw a graph with two sets of data.

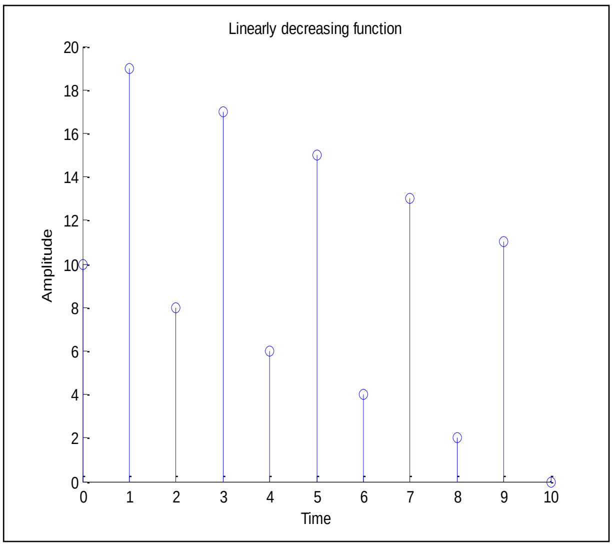


Figure 3: Graph of a linearly decreasing function

In this case we see a large variation between the expected value and the obtained value. But a little bit modification for the decreasing function can make the result perfect. The modified equation is:

$$f(x) = g(u) = u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots + \frac{u(u-1)(u-2)\dots\dots(u-n+1)}{n!}\Delta^n y_0 \quad \dots\dots\dots\dots (5)$$

Using equation (5) the calculated values are shown in Table 8:

| Time (T) | Amplitude(x) |
|---|---|
| 1 | 9 |
| 3 | 7 |
| 5 | 5 |
| 7 | 3 |
| 9 | 1 |

Table 8: Calculated values of a linearly decreasing function

Now, the result can be observed by the following graph:

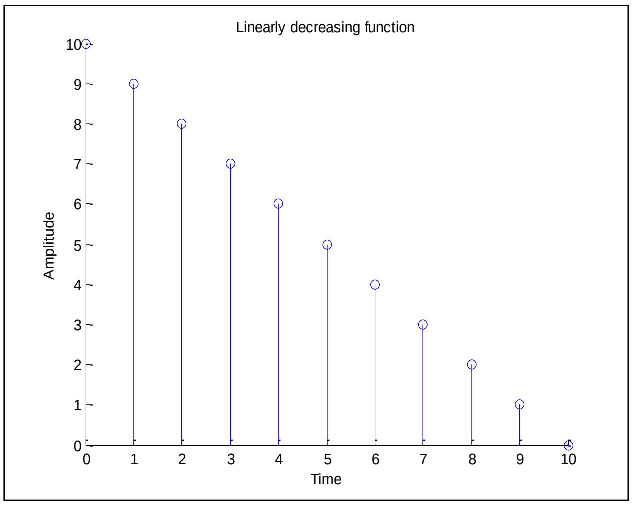


Figure 4: Graph of a linearly decreasing function

Finally I've considered a sinusoid. To reduce the calculation we only calculate the points for one quarter of a cycle. The obtained values are shown below:

| T | x | $\Delta x$ | $\Delta^2 x$ | $\Delta^3 x$ | $\Delta^4 x$ | $\Delta^5 x$ | $\Delta^6 x$ | $\Delta^7 x$ | $\Delta^8 x$ | $\Delta^9 x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | -0.309 | 0.0302 | 0.0274 | -0.0059 | -0.0015 | -0.0006 | 0.003 | -0.006 | 0.012 |
| 1.1 | -0.309 | -0.2788 | 0.0576 | 0.0215 | -0.0074 | -0.0021 | 0.0024 | -0.003 | 0.006 | |
| 1.2 | -0.5878 | -0.2212 | 0.0791 | 0.0141 | -0.0095 | 0.0003 | -0.0006 | 0.003 | | |
| 1.3 | -0.809 | -0.1421 | 0.0932 | 0.0046 | -0.0092 | -0.0003 | 0.0024 | | | |
| 1.4 | -0.9511 | -0.0489 | 0.0978 | -0.0046 | -0.0095 | 0.0021 | | | | |
| 1.5 | -1 | 0.0489 | 0.0932 | -0.0141 | -0.0074 | | | | | |
| 1.6 | -0.9511 | 0.1421 | 0.0791 | -0.0215 | | | | | | |
| 1.7 | -0.809 | 0.2212 | 0.0576 | | | | | | | |
| 1.8 | -0.5878 | 0.2788 | | | | | | | | |
| 1.9 | -0.309 | | | | | | | | | |

Table 9: Forward Difference Table for a quarter of a cycle of a sinusoid

| Time (T) | Amplitude (x) |
|---|---|
| 1.05 | 1.000204 |
| 1.15 | -0.39804 |
| 1.25 | -0.3999 |
| 1.35 | -0.08203 |
| 1.45 | 0.497075 |
| 1.55 | 1.151958 |
| 1.65 | 1.761094 |
| 1.75 | 3.063 |
| 1.85 | 2.3698 |

Table 10: Calculated values of the sinusoid using equation (1)

Now from the graph it is seen that the variation between actual values and the obtained values are large.

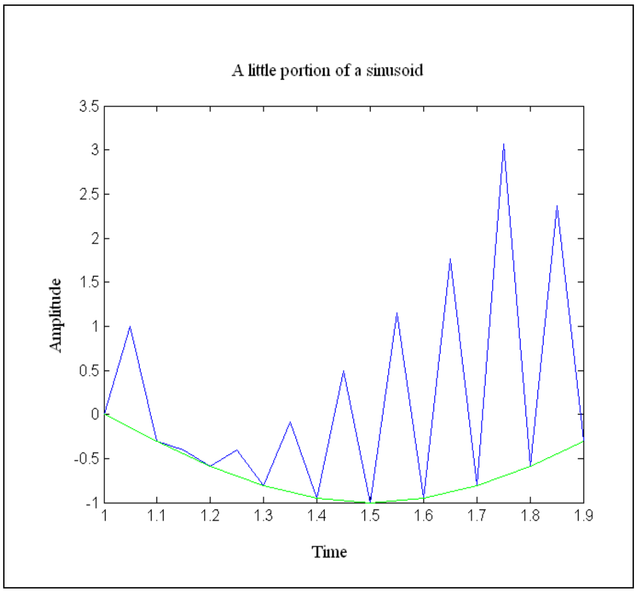


Figure 4: Comparison between the actual values and the calculated values of a sinusoid

From here we observe that to construct the signal in this case this method shows the worst performance.

## 4. Experimental results

From the analysis the limitation we've founded can expressed as follows:

i) For a step function it gives 100% accuracy

ii) For a linearly increased function like ramp function it gives 100% accuracy but if the function is decreasing (linearly) it can't give the right answers. A little bit modification is needed to get 100% accuracy.

iii) The worst case happens when the function can't be defined as a increasing or decreasing function but a combination.

Finally we can say that It gives the best result when the function is constant or increased linearly.

## 5. Conclusion

In this paper, according to the analysis the performance of Newton interpolation formula on different types of functions is presented. Experimental results show that for reconstructing a signal (e.g. image resizing) it works better for the area where signal values are relatively constant or increasing. In conclusion, it can be said that this formula is designed for a function whose value will increase or remain constant with the independent variable.

## 6. References

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